Duration : 144 minutes

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Linear Algebra Exam

Common part

Fall 2019

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2019.

Notation

– For a matrix A, a_{ij} denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x}$, x_i denotes the i-the coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– The inner product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : For a real parameter a, and

$$p_1(t) = a + 4t - 5t^2\,, \qquad p_2(t) = 4 + at - 5t^2\,, \qquad p_3(t) = 4 - 5t + at^2\,,$$

the polynomials p_1, p_2 and p_3 are linearly dependent if and only if

- [ ] $a \notin \{-5, 1, 4\}$
- [x] $a \in \{-5, 1, 4\}$
- [ ] $a \in \{-5, -1, 4\}$
- [ ] $a \notin \{-5, -1, 4\}$

Question 2 : Let

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

Then a basis for $\{\boldsymbol{x} \in \mathbb{R}^3 \mid A\boldsymbol{x} = 5\boldsymbol{x}\}$ is given by

- [ ] $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$
- [ ] $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$
- [x] $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$
- [ ] $\left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

Question 3 : Let A be a diagonalisable $n \times n$ matrix.
If all the eigenvalues of A are non-zero, it is always true that

- [ ] A^T and A^{-1} are not necessarily diagonalisable
- [x] A^T and A^{-1} are diagonalisable
- [ ] A^T is diagonalisable, but A^{-1} is not necessarily diagonalisable
- [ ] A^{-1} is diagonalisable, but A^T is not necessarily diagonalisable

Question 4 : Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 3 & 6 \\ 0 & -5 & 15 & 18 \\ 0 & 2 & -6 & -6 \end{pmatrix}.$$

The eigenvalues of A are

- ☐ 3, 5, 0 and 2
- ■ 2, 6, 0
- ☐ -1, 15, -6 and 2
- ☐ -2, 2 and 0

Question 5 : Let A be a square $n \times n$ matrix.
If A is orthogonal, which of the following statements **IS NOT** necessarily true?

- ☐ For all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, $A\boldsymbol{v}$ is orthogonal to $A\boldsymbol{w}$ if and only if $\boldsymbol{v}$ is orthogonal to $\boldsymbol{w}$
- ☐ $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$ for all $1 \leq i, j \leq n$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are the columns of A and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ are the columns of A^T
- ■ $\det A = 1$
- ☐ A^T is orthogonal

Question 6 : Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear mapping defined by

$$T\left(a + bt + ct^2 + dt^3\right) = (a-d) + (b+c)t + (c-b)t^2 + (a+d)t^3.$$

The matrix M for T with respect to the basis $\mathcal{B} = \left\{1+t^3,\ 1-t^3,\ -t+t^2,\ t+t^2\right\}$, i.e. $\left[T(p)\right]_{\mathcal{B}} = M\left[p\right]_{\mathcal{B}}$ for all $p \in \mathbb{P}_3$ is

- ☐ $M = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$
- ☐ $M = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$
- ■ $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
- ☐ $M = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$

Question 7 : For a real number α, and

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & \alpha & 1 \\ -1 & 1 & 1 & \alpha \end{pmatrix},$$

the matrix A is invertible if and only if

- ☐ $\alpha \in \{3, -1\}$
- ☐ $\alpha \in \{-3, 1\}$
- ■ $\alpha \notin \{-3, 1\}$
- ☐ $\alpha \notin \{3, -1\}$

Question 8 : Let A be any $m \times n$ matrix with $m < n$. Then it is always true that

- ■ $A\boldsymbol{x} = A\boldsymbol{c}$ has an infinite number of solutions for all choices of $\boldsymbol{c} \in \mathbb{R}^n$
- ☐ $A\boldsymbol{x} = A\boldsymbol{c}$ has a unique solution for all choices of $\boldsymbol{c} \in \mathbb{R}^n$
- ☐ $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for all choices of $\boldsymbol{b} \in \mathbb{R}^m$
- ☐ $A^T\boldsymbol{y} = \boldsymbol{0}$ has a unique solution

Question 9 : Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$$T(\boldsymbol{e}_1) = 6\boldsymbol{e}_1 + 12\boldsymbol{e}_2 - 3\boldsymbol{e}_3, \qquad T(\boldsymbol{e}_2) = 2\boldsymbol{e}_1 + 4\boldsymbol{e}_2 - \boldsymbol{e}_3,$$

$$T(\boldsymbol{e}_3) = 8\boldsymbol{e}_1 + 12\boldsymbol{e}_2 - 8\boldsymbol{e}_3, \qquad T(\boldsymbol{e}_4) = 8\boldsymbol{e}_1 + 10\boldsymbol{e}_2 - 10\boldsymbol{e}_3,$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are the usual canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Then the range (or image) $\operatorname{Im} T$ of T is

- ☐ $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}\right\}$
- ☐ $\text{Span}\left\{\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\right\}$
- ☐ $\text{Span}\left\{\begin{pmatrix} 6 \\ 12 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}\right\}$
- ■ $\text{Span}\left\{\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\right\}$

Question 10 : As $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$ is symmetric, it is orthogonally diagonalisable, i.e. it can be written as $A = QDQ^T$, with Q an orthogonal matrix and D a diagonal matrix.

If $d_{11} > 0$, then one possible choice for Q is

- ☐ $Q = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$
- ☐ $Q = \begin{pmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{pmatrix}$
- ☐ $Q = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$
- ■ $Q = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$

Question 11 : Let A be a $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$.
Let $\boldsymbol{w} \in \mathbb{R}^n$ be a solution of the linear system $(A^T A)\boldsymbol{w} = A^T \boldsymbol{b}$. Then it is always true that

- ☐ $\boldsymbol{w}$ is a solution of the linear system $A\boldsymbol{w} = \boldsymbol{b}$
- ☐ the matrix $A^T A$ is invertible
- ☐ $\|\boldsymbol{b} - A\boldsymbol{w}\| \geq \|\boldsymbol{b} - A\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$
- ■ $\|\boldsymbol{b} - A\boldsymbol{w}\| \leq \|\boldsymbol{b} - A\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$

Question 12 : Let $a_0, a_1, a_2 \in \mathbb{R}$. Among the following four sub-sets of the vector space $\mathbb{P}_2$:

$$\begin{aligned}
\mathcal{E}_1 &= \{a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2 \mid a_1 = 0\}, \\
\mathcal{E}_2 &= \{a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2 \mid a_2 = a_0 + a_1\}, \\
\mathcal{E}_3 &= \{a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2 \mid a_1 = a_2 + 3\}, \\
\mathcal{E}_4 &= \{a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2 \mid a_0^2 = a_1^2\},
\end{aligned}$$

how many are sub-spaces?

- ☐ 1
- ■ 2
- ☐ 3
- ☐ 4

Question 13 : Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ 8x_1 + x_2 \end{pmatrix},$$

and let M denote the matrix for T relative to a basis $\mathcal{B}$, that is M is such that $[T(\boldsymbol{v})]_{\mathcal{B}} = M[\boldsymbol{v}]_{\mathcal{B}}$ for all $\boldsymbol{v} \in \mathbb{R}^2$. If the basis $\mathcal{B} = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right\}$, and $M = \begin{pmatrix} -5 & 10 \\ -2 & 7 \end{pmatrix}$, then

- ■ $b_1 = 2, \quad b_2 = 1$
- ☐ $b_1 = 1, \quad b_2 = -2$
- ☐ $b_1 = 1, \quad b_2 = 2$
- ☐ $b_1 = -2, \quad b_2 = 1$

Question 14 : For

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & -2 & -1 \end{pmatrix},$$

the entries in the inverse matrix $C = A^{-1}$ satisfy

- ■ $c_{11} = -1$ and $c_{32} = -1$
- □ $c_{21} = -1$ and $c_{13} = 0$
- □ $c_{22} = -1$ and $c_{13} = -1$
- □ $c_{12} = -1$ and $c_{33} = 0$

Question 15 : Let A be an $m \times n$ matrix, and let W be the subspace of $\mathbb{R}^m$ defined by

$$W = \{\boldsymbol{w} \in \mathbb{R}^m \mid \text{there exists a } \boldsymbol{v} \in \mathbb{R}^n \text{ with } A\boldsymbol{v} = \boldsymbol{w}\}\ .$$

If $\dim(W) = k$, then

- □ $\dim(\mathrm{Nul}\, A^T) = n - k$
- □ $\dim(\mathrm{Nul}\, A^T) = k$
- □ $\dim(\mathrm{Nul}\, A^T) = \min(m, n) - k$
- ■ $\dim(\mathrm{Nul}\, A^T) = m - k$

Question 16 : Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear mapping defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 - 3x_3 + 4x_4 \\ 2x_1 + x_2 - x_4 \\ 3x_1 + x_2 + x_3 - 3x_4 \\ x_2 - 2x_3 + 3x_4 \end{pmatrix}.$$

Then the nullspace (or kernel) of T is

- □ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ -1 \end{pmatrix}\right\}$
- ■ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right\}$
- □ $\mathrm{Span}\left\{\begin{pmatrix} 3 \\ -4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}\right\}$
- □ $\mathrm{Span}\left\{\begin{pmatrix} -1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}\right\}$

Question 17 : For a real parameter h,

$$A = \begin{pmatrix} 1 & -4 & -3 \\ -1 & 12 & 5 \\ -1 & 4h+4 & h+3 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ h-3 \\ 2 \end{pmatrix},$$

the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has an infinite number of solutions if and only if

- ☐ $h \in \{4, 1\}$
- ☐ $h \in \{-4, 1\}$
- ☐ $h \in \{-4, -1\}$
- ■ $h \in \{4, -1\}$

Question 18 : Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$$

Then the least squares solution $\widehat{\boldsymbol{x}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$ of $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

- ■ $\widehat{x}_1 = 10/7,\qquad \widehat{x}_2 = 12/7$
- ☐ $\widehat{x}_1 = -10/7,\qquad \widehat{x}_2 = 12/7$
- ☐ $\widehat{x}_1 = 12/7,\qquad \widehat{x}_2 = 10/7$
- ☐ $\widehat{x}_1 = 12/7,\qquad \widehat{x}_2 = -10/7$

Question 19 : For A any $n \times n$ matrix, let

$$k = \det\left((A + I_n)^2 - (A - I_n)^2\right).$$

Then

- ☐ $k = 4\det(A)$
- ☐ $k = 2^n \det(A)$
- ☐ $k = 2\det(A)$
- ■ $k = 4^n \det(A)$

Question 20 : Let

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 9 \\ 0 \\ -18 \end{pmatrix} \quad \text{and} \quad W = \text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 6 \\ 2 \end{pmatrix} \right\}.$$

Then the orthogonal projection of $\boldsymbol{v}$ on W is

- ☐ $\begin{pmatrix} 8 \\ 1 \\ 0 \\ -14 \end{pmatrix}$
- ☐ $\begin{pmatrix} -12 \\ 12 \\ -6 \\ -6 \end{pmatrix}$
- ■ $\begin{pmatrix} -8 \\ 8 \\ 0 \\ -4 \end{pmatrix}$
- ☐ $\begin{pmatrix} -360 \\ 360 \\ -432 \\ -180 \end{pmatrix}$

Question 21 : Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^3$. Then the change of coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$, i.e. the matrix such that $[\boldsymbol{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\boldsymbol{x}]_{\mathcal{B}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$, is

- ☐ $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
- ■ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$
- ☐ $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix}$

Question 22 : Let A and B be two square $n \times n$ matrices and let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ be a basis of $\mathbb{R}^n$ formed by eigenvectors of A.

If the $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are also eigenvectors of the matrix AB, then it is always true that

- ☐ if B is invertible, then B is diagonalisable
- ☐ if A is invertible, then $AB \neq BA$
- ■ if A is invertible, then B is diagonalisable
- ☐ the determinant of B is non-zero